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Orthogonality of matrices ☆

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Abstract

Let X be a real finite-dimensional normed space with unit sphere S_X and let $\mathscr{L}(X)$ be the space of linear operators from X into itself. It is proved that X is an inner product space if and only if for $A, C \in \mathscr{L}(X)$

$$A \perp C \Leftrightarrow \exists u \in S_X : \|A\| = \|Au\|, \quad Au \perp Cu,$$

where $\perp$ denotes Birkhoff orthogonality.

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1. Introduction

Let X be a normed space over $\mathbb{K}$ ($\mathbb{R}$ or $\mathbb{C}$) with unit sphere S_X and let $\mathscr{L}(X)$ be the space of continuous linear operators from X into itself endowed with the usual induced norm.

$A \in \mathscr{L}(X)$ is said to be orthogonal (in the sense of Birkhoff) to $C \in \mathscr{L}(X)$, in short $A \perp C$, when $\|A\| \leqslant \|A + \lambda C\|$ for every $\lambda \in \mathbb{K}$. The concept of Birkhoff orthogonality for vectors in X is defined in the same way as for operators in $\mathscr{L}(X)$ (see [4,5]).

It is immediate that if there is $u \in S_X$ such that $\|A\| = \|Au\|$ and $Au \perp Cu$, then $A \perp C$. It suffices to consider that

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$$\|A\| = \|Au\| \leqslant \|Au + \lambda Cu\| \leqslant \sup_{v \in S_X} \|Av + \lambda Cv\| = \|A + \lambda C\|, \quad (\lambda \in \mathbb{K}).$$

Bhatia and Šemrl [3] proved that if X is a real or complex finite-dimensional inner product space then the converse of the above proposition, is also true i.e.,

$$A \perp C \Rightarrow \exists u \in S_X : \|A\| = \|Au\|, \quad Au \perp Cu. \tag{$*$}$$

In the same paper it is conjectured that $(*)$ is valid for any finite-dimensional normed space X.

Chi-Kwong Li and Schneider [6] give a counterexample to the above conjecture. They show that it does not hold for X the space ℓ_p^n, with $p \neq 2$.

In this paper it is proved that actually there are counterexamples in every real finite-dimensional normed space whose norm is not induced by an inner product. I.e., the property $(*)$ is characteristic of real inner product spaces of finite dimension.

Two facts are worth recalling:

First, the central role of Birkhoff orthogonality in approximation theory, typified by the fact that C_0 is a best approximation of $A \in \mathscr{L}(X)$ from a linear subspace $\mathscr{M}$ of $\mathscr{L}(X)$ if and only if C_0 is a Birkhoff orthogonal projection of A onto $\mathscr{M}$, i.e., $A - C_0 \perp \mathscr{M}$.

Second, that $(*)$ fails in infinite-dimensional inner product spaces. For example, the operator $A : (x_1, x_2, x_3, \ldots) \in \ell_2 \to \left(\frac{1}{2}x_1, \frac{2}{3}x_2, \frac{3}{4}x_3, \ldots\right)$ is Birkhoff-orthogonal to $C : (x_1, x_2, x_3, \ldots) \in \ell_2 \to (x_1, 0, 0, \ldots)$, but there does not exist $u = (u_1, u_2, u_3, \ldots) \in S_{\ell_2}$ such that $\|A\| = \|Au\|$.

2. Two-dimensional case

Suppose that X is a real 2-dimensional normed space, i.e., the space $\mathbb{R}^2$ endowed with a norm with closed unit ball B_X and unit sphere S_X.

Lemma 2.1. *For any linearly independent* $u, v \in S_X$ *and any* $0 < \rho < 1$ *there exists* $A \in \mathscr{L}(X)$ *such that*

(1) $Au = \rho v$,
(2) $A(S_X) \subset B_X$,
(3) $A(S_X) \cap S_X \neq \emptyset$,
(4) $A(S_X) \cap S_X$ *has at least two connected components.*

Proof. Since there is a linear operator that carries u and v into $(1, 0)$ and $(0, 1)$, respectively, we can suppose (in order to simplify calculations) that $u = (1, 0)$ and $v = (0, 1)$.

(1) From

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix} \begin{pmatrix} 1 \\ 0 \end{pmatrix} = \begin{pmatrix} 0 \\ \rho \end{pmatrix},$$

it follows that

$$A = \begin{pmatrix} 0 & b \\ \rho & d \end{pmatrix}.$$

(2) Let A be as above. On the one hand, it is obvious that for each $0 < \rho < 1$ there is a $\sigma > 0$ such that

$$\{(x, y) : |x| \leqslant \sigma, |y| \leqslant \rho\} \subset B_X.$$

On the other hand, since B_X is convex and bounded and the points $(1, 0)$ and $(0, 1)$ are on its boundary, there is a straight line $x + ty = 1$, with $|t| \leqslant 1$, that supports B_X at $u = (1, 0)$, and another straight line $y = r$, with $r > 0$, that also supports B_X. Hence

$$(x, y) \in S_X \Rightarrow |x + ty| \leqslant 1, \quad |y| \leqslant r,$$

and, for $b = \frac{\sigma}{r}$ and $d = \rho t$, we obtain that the image by A of any point $(x, y) \in S_X$,

$$\begin{pmatrix} 0 & \frac{\sigma}{r} \\ \rho & \rho t \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} \frac{\sigma}{r} y \\ \rho x + \rho t y \end{pmatrix},$$

belongs to the rectangle $\{(x, y) : |x| \leqslant \sigma, |y| \leqslant \rho\}$. I.e., the set of operators A such that $Au = \rho v$ and $A(S_X) \subset B_X$ is non-empty.

(3) Suppose now that the operators

$$A = \begin{pmatrix} 0 & b \\ \rho & d \end{pmatrix}$$

are such that $Au = \rho v$ and $A(S_X) \subset B_X$. By continuity there exist $b, d \in \mathbb{R}$ such that the area enclosed by $A(S_X)$ is maximal among the areas enclosed by all $C(S_X)$, with $C \in \mathscr{L}(X), Cu = \rho v$, $C(S_X) \subset B_X$.

If $A(S_X) \cap S_X = \emptyset$, also by continuity there exists $a > 1$ such that for

$$D = \begin{pmatrix} a & 0 \\ 0 & 1 \end{pmatrix},$$

$DAu = \rho v$, $DA(S_X) \subset B_X$, and

$$\text{area}[DA(B_X)] = \det D \, \text{area}[A(B_X)] > \text{area}[A(B_X)],$$

which is contradictory.

Then $A(S_X) \cap S_X \neq \emptyset$, as we wished to show.

(4) It suffices to notice that, for the above A, $\pm\rho v \in A(S_X)$. Hence $\pm v \notin A(S_X) \cap S_X$. □

In what follows we shall denote by $A_{u,\rho v}$ the operator of Lemma 2.1, i.e., an operator such that the area enclosed by $A_{u,\rho v}(S_X)$ is maximal among the areas enclosed by all $C(S_X)$, with $C \in \mathscr{L}(X)$, $Cu = \rho v$, $C(S_X) \subset B_X$.

Remark 2.2. Since $A_{u,\rho v}(S_X) \cap S_X$ is symmetric with respect to the origin, the number of its connected components is even, if finite.

Remark 2.3. The operator $A_{u,\rho v}$ is not necessarily unique. For example, when $X = \ell^2_\infty$, $u = \left(1, \frac{1}{2}\right)$, and $\rho v = \left(0, \frac{1}{2}\right)$, $A_{u,\rho v}(S_X)$ may be either the centrally symmetric parallelogram with $\left(\frac{1}{3}, 1\right)$ and $(1, 1)$ as vertices, or the centrally symmetric parallelogram with $\left(-\frac{1}{3}, 1\right)$ and $(-1, 1)$ as vertices.

Lemma 2.4. *If S_X is not an ellipse (X is not an inner product space), then there is an $A \in \mathscr{L}(X)$ such that $A(S_X) \subset B_X$ and $A(S_X) \cap S_X$ has at least four connected components.*

Proof. (1) If for some linearly independent $u, v \in S_X$ and some $0 < \rho < 1$ the set $A_{u,\rho v}(S_X) \cap S_X$ has more than two connected components, we have finished by taking $A = A_{u,\rho v}$.

(2) If for some linearly independent $u, v \in S_X$ and some $0 < \rho < 1$ the set $A_{u,\rho v}(S_X) \cap S_X$ has two connected components, we will prove that

$$\{^{\perp}v\} \cap A_{u,\rho v}(S_X) \cap S_X \neq \emptyset,$$

where $\{^{\perp}v\} = \{w \in S_X : w \perp v\}$, using an argument similar to that in J. Oman [7, Theorem 1] which shows an analogous question with ellipses that are inscribed in S_X.

Indeed, suppose on the contrary that $\{^{\perp}v\} \cap A_{u,\rho v}(S_X) \cap S_X = \emptyset$. Without loss of generality we may assume that

$$\rho v = (0, \rho), \quad \{^{\perp}v\} = \{(\pm 1, y) : |y| \leqslant \sigma\},$$

where $\sigma \geqslant 0$, and that one of the connected components of the set $A_{u,\rho v}(S_X) \cap S_X$ (the other is its symmetric image about the origin) is formed by points (x, y) such that, $0 < \alpha \leqslant x \leqslant \beta < 1$ and $\sigma < \gamma \leqslant y$.

Then, for

$$C = \begin{pmatrix} 1 & 0 \\ -c & 1 \end{pmatrix},$$

with $c > 0$ sufficiently small,

$$\begin{aligned} &CA_{u,\rho v}u = \rho v, \quad CA_{u,\rho v}(S_X) \subset B_X, \quad CA_{u,\rho v}(S_X) \cap S_X = \emptyset, \\ &\text{area}[CA_{u,\rho v}(B_X)] = \det C \,\text{area}[A_{u,\rho v}(B_X)] = \text{area}[A_{u,\rho v}(B_X)], \end{aligned}$$

which is absurd.

(3) Suppose now that for some linearly independent $u, v \in S_X$ and any $0 < \rho < 1$ the set $A_{u,\rho v}(S_X) \cap S_X$ has two connected components, and hence that $\{^{\perp}v\} \cap A_{u,\rho v}(S_X) \cap S_X \neq \emptyset$.

Since we are in finite dimensions there exists a sequence

$$(\rho_n, A_{u,\rho_n v}, (^{\perp}v)_n),$$

where $0 < \rho_n < 1$ and $(^{\perp}v)_n \in \{^{\perp}v\} \cap A_{u,\rho_n v}(S_X) \cap S_X$, that converges to $(1, A_{u,v}, (^{\perp}v)_0)$ with $(^{\perp}v)_0 \in \{^{\perp}v\} \cap A_{u,v}(S_X) \cap S_X$.

By continuity

$$A_{u,v}u = v, \quad A_{u,v}(S_X) \subset B_X, \quad (^{\perp}v)_0, v \in A_{u,v}(S_X) \cap S_X.$$

If $(^{\perp}v)_0, -(^{\perp}v)_0$, and v are in different connected components of $A_{u,v}(S_X) \cap S_X$, we have finished by taking $A = A_{u,v}$.

(4) Finally, suppose that we are in the worst case that, for every linearly independent $u, v \in S_X$, the set $A_{u,v}(S_X) \cap S_X$ has at most two connected components. I.e., either $A_{u,v}(S_X) = S_X$ or $A_{u,v}(S_X) \cap S_X$ has two connected components. In both cases (see (3)), either the shorter arc of S_X that goes from $(^{\perp}v)_0$ to v, or the shorter arc from $-(^{\perp}v)_0$ to v, is contained in $A_{u,v}(S_X) \cap S_X$.

The above is not possible when either u is an inner point of a segment of S_X but v is not, or u is a smooth point of S_X but v is not. In other words, X must be rotund and smooth. Hence [5, pp. 274, 275], for any $v \in S_X$ there are unique $^{\perp}v, v^{\perp} \in S_X$ such that $^{\perp}v \perp v$, $v \perp v^{\perp}$, and $^{\perp}v \prec v \prec v^{\perp}$, where $\prec$ is an orientation of the plane X.

Summarizing, X is rotund and smooth, and for any linearly independent $u, v \in S_X$ there exists $A_{u,v} \in \mathscr{L}(X)$ such that

$$A_{u,v}u = v, \quad A_{u,v}(S_X) \subset B_X, \quad {}^{\perp}v, v \in A_{u,v}(S_X) \cap S_X.$$

Let $w \in S_X$ satisfying $A_{u,v}w = {}^{\perp}v$. Then, for every $\lambda \in \mathbb{R}$

$$\|w\| = \|^{\perp}v\| \leqslant \|^{\perp}v + \lambda v\| = \|A_{u,v}(w + \lambda u)\| \leqslant \|w + \lambda u\|,$$

i.e., $w = \pm {}^{\perp}u$.

Let $v \in S_X$ be such that $\|{}^{\perp}v + v\| + \|{}^{\perp}v - v\|$ is maximum. Then, for any $u \in S_X$ we have that

$$\|{}^{\perp}v \pm v\| = \|A_{u,v}{}^{\perp}u \pm A_{u,v}u\| \leqslant \|{}^{\perp}u \pm u\|,$$

and hence, either

$$\|{}^{\perp}u + u\| = \|{}^{\perp}v + v\|, \quad \|{}^{\perp}u - u\| = \|{}^{\perp}v - v\|,$$

or

$$\|{}^{\perp}u + u\| = \|{}^{\perp}v - v\|, \quad \|{}^{\perp}u - u\| = \|{}^{\perp}v + v\|.$$

It is known [2] (see e.g., [1, p. 79]) that

$$\{\|{}^{\perp}u + u\| : u \in S_X\}, \quad \{\|{}^{\perp}u - u\| : u \in S_X\}$$

are either $\{\sqrt{2}\}$ (and X is an inner product space) or an interval (neighbourhood of $\sqrt{2}$).

Therefore, the only possibility is

$$\|{}^{\perp}u + u\| = \|{}^{\perp}u - u\| = \sqrt{2} \quad \text{for every } u \in S_X,$$

which is a characteristic property of inner product spaces [1, p. 79], in contradiction with the hypothesis that S_X is not an ellipse. $\square$

For $x = (x_1, x_2)$ and $y = (y_1, y_2)$ in X we shall use the notation $x \prec y$ when x precedes y in the positive orientation of X, i.e.,

$$x \wedge y = x_1 y_2 - x_2 y_1 > 0.$$

Since for $A \in \mathscr{L}(X)$, $(Ax) \wedge (Ay) = (\det A)(x \wedge y)$, A preserves the orientation if and only if $\det A > 0$.

Proposition 2.5. *If S_X is not an ellipse (X is not an inner product space), then there exist $A, C \in \mathscr{L}(X)$ such that $A \perp C$, but there does not exist $u \in S_X$ such that $\|A\| = \|Au\|$ and $Au \perp Cu$.*

Proof. Let A be as in Lemma 2.4 and let $\prec$ be the positive orientation of X. Since $A(S_X) \subset B_X$, and $A(S_X) \cap S_X$ has at least four connected components, there are $u_1, u_2, u_3, u_4 \in S_X$ such that

$$u_1 \preceq u_2 \prec u_3 \preceq u_4 \prec -u_1, \quad Au_1, Au_2, Au_3, Au_4 \in S_X,$$

and

$$u \in S_X, u_2 \prec u \prec u_3 \text{ or } u_4 \prec u \prec -u_1 \quad \text{implies} \quad Au \in \operatorname{int} B_X.$$

Define $C \in \mathscr{L}(X)$ by (see Fig. 1)

$$Cu_1 = Au_4 + Au_1, \quad Cu_3 = Au_2 - Au_3.$$

Since the open segments $]Au_4, -Au_1[$ and $]Au_2, Au_3[$ are in $\operatorname{int} B_X$, for every $\lambda > 0$,

$$\|A\| = \|Au_1\| < \|Au_1 + \lambda(Au_4 + Au_1)\| = \|(A + \lambda C)u_1\| \leqslant \|A + \lambda C\|,$$
$$\|A\| = \|Au_3\| < \|Au_3 - \lambda(Au_2 - Au_3)\| = \|(A - \lambda C)u_3\| \leqslant \|A - \lambda C\|,$$

i.e., $A \perp C$.

To see that there does not exist $u \in S_X$ such that $\|A\| = \|Au\|$ and $Au \perp Cu$, first note that if $u \in S_X$ satisfies $\|A\| = \|Au\|$ then it is either between u_1 and u_2, or between u_3 and u_4 (or their symmetric images about the origin).

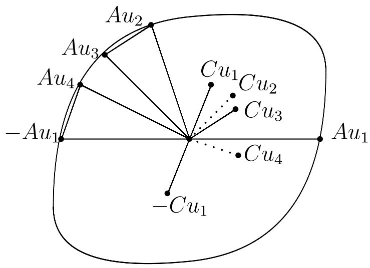

Fig. 1. Proof of $A \perp C$.

Suppose (other cases are analogous) that

$$u_1 \preceq u \preceq u_2$$

and that

$$Au_1 \preceq Au_2 \prec Au_3 \preceq Au_4 \prec -Au_1.$$

Then

$$(\det C)(u_3 \wedge u_1) = Cu_3 \wedge Cu_1 = (Au_2 - Au_3) \wedge (Au_4 + Au_1) > 0.$$

Hence $\det C < 0$ and C reverses the orientation.

Therefore

$$Au_2 - Au_3 = Cu_3 \prec Cu_2 \preceq Cu \preceq Cu_1 = Au_4 + Au_1.$$

For $v \in S_X$, let $(Av)^{\perp}$ denote any point $w \in S_X$ such that $Av \prec w$ and $Av \perp w$.

Since the segments $[Au_4, -Au_1]$ and $[Au_2, Au_3]$ are chords of S_X, it follows from $Au_1 \preceq Au \preceq Au_2$ (see Fig. 1) that

$$Au_4 + Au_1 \prec (Au_1)^{\perp} \preceq (Au)^{\perp} \preceq (Au_2)^{\perp} \prec -Au_2 + Au_3,$$

and thus

$$Au_2 - Au_3 \prec Cu \prec Au_4 + Au_1 \prec (Au)^{\perp} \prec -Au_2 + Au_3,$$

shows that Au cannot be orthogonal to Cu. $\square$

3. *n*-dimensional case

Let X be a real normed space of dimension $n \geqslant 3$. For $x \in X$ we shall write $Jx = \{f \in S_{X^*} : f(x) = \|x\|\}$, where X^* is the topological dual of X.

A well known theorem of Brunn, Blaschke, and Kakutani (see, e.g., [1, p. 99]) says that if X is not an inner product space then there exists a 2-dimensional subspace Y such that the norm of every continuous linear projection $A : X \to Y$ is greater than 1. I.e., $B_Y = B_X \cap Y \subset A(B_X)$ and $B_Y = B_X \cap Y \neq A(B_X)$.

Lemma 3.1. *If X is not an inner product space, then there exist a 2-dimensional subspace Y and a linear projection $A : X \to Y$ with $\|A\| > 1$, such that $\{Ax : x \in S_X, \|Ax\| = \|A\|\}$ has at least four connected components.*

Proof. Let Y be a 2-dimensional subspace of X such that the norm of every linear projection of X onto Y is greater than 1, and let $u \in S_Y$ be an exposed point of S_Y (relative to Y), i.e., a point for which there exists $f \in Ju$ such that $f \notin Jv$ for every $v \in S_Y$ other than u.

For $v \in S_Y$, $v \neq \pm u$, and $g \in Jv$, $X = Y \oplus (\ker f \cap \ker g)$. Then the linear projection

$$A_{vg} : y + z \in Y \oplus (\ker f \cap \ker g) \to y \in Y$$

is such that

$$\|A_{vg}u\| = \|A_{vg}v\| = 1 < \|A_{vg}\|.$$

Let $\prec$ be the positive orientation of the plane Y. First we prove that if $v \in S_Y$ is sufficiently close to u and $u \prec v$, then A_{vg} does not attain its norm at x, provided that $g \in Jv$, $x \in S_X$, and $u \prec A_{vg}x \prec v$.

Otherwise, there would exist sequences (v_n) in S_Y and (x_n) in S_X such that

$$v_n \to u, u \prec \cdots \prec v_3 \prec v_2 \prec v_1 \prec -u, \quad \text{(i.e. } (v_n) \downarrow u),$$
$$0 < f(v_1) < f(v_2) < f(v_3) < \cdots < 1,$$

and

$$1 < \|A_{v_n g_n}\| = \|A_{v_n g_n}x_n\|, \quad u \prec A_{v_n g_n}x_n \prec v_n$$

for some $g_n \in Jv_n$. (Note that $v_n \to u$ implies $f(v_n) \to f(u) = 1$. So we may suppose that for every n, $0 < f(v_n)$.)

Since (see Fig. 2) $B_X \subset \{z \in X : f(z) \leqslant 1\}$, $f(A_{v_n g_n}x_n) \leqslant 1$. By convexity of B_Y

$$f(v_n) \leqslant f(\|A_{v_n g_n}x_n\|v_n) = \|A_{v_n g_n}x_n\| f(v_n) \leqslant 1.$$

Then

$$1 < \|A_{v_n g_n}\| = \|A_{v_n g_n}x_n\| \leqslant \frac{1}{f(v_n)},$$

and hence

$$\lim_{n\to\infty} \|A_{v_n g_n}\| = \lim_{n\to\infty} \|A_{v_n g_n}x_n\| = \frac{1}{\lim_{n\to\infty} f(v_n)} = 1.$$

It follows (we are in finite dimensions) that the sequence $(A_{v_n g_n})$ has a subsequence that converges to a norm-1 linear projection of X onto Y, in contradiction with our hypothesis about Y.

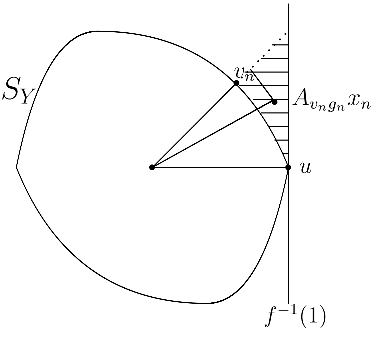

Fig. 2. Proof of $f(v_n) \leqslant 1$.

Similarly, we can prove that if $v \in S_Y$ is sufficiently close to $-u$ and $v \prec -u$, then A_{vg} does not attain its norm at x, provided that $g \in Jv$, $x \in S_X$, and $v \prec A_{vg}x \prec -u$. Since X is finite-dimensional, A_{vg} attains its norm at some $x \in S_X$ such that $u \prec A_{vg}x \prec v$. Note that

$$1 = \|A_{vg}u\| = \|A_{vg}v\| < \|A_{vg}\| = \|A_{vg}x\|,$$

then $A_{vg}x \neq u, v$.

Therefore the set of $v \in S_Y$ that satisfy the property

$$\exists g \in Jv, \exists x \in S_X : \|A_{vg}\| = \|A_{vg}x\|, \quad u \prec A_{vg}x \prec v$$

is nonempty. Let $\bar{v}$ be the infimum (in the order $\prec$ of S_Y) of such a set.

Then $u \prec \bar{v} \prec -u$. We shall prove in three steps that there exist $\bar{g} \in J\bar{v}$ and $\bar{x}, \bar{y} \in S_X$ such that

$$1 = \|A_{\bar{v}\bar{g}}u\| = \|A_{\bar{v}\bar{g}}\bar{v}\| < \|A_{\bar{v}\bar{g}}\bar{x}\| = \|A_{\bar{v}\bar{g}}\bar{y}\| = \|A_{\bar{v}\bar{g}}\|, \quad u \prec A_{\bar{v}\bar{g}}\bar{x} \prec \bar{v} \prec A_{\bar{v}\bar{g}}\bar{y} \prec -u.$$

(1) From the definition of $\bar{v}$, there follows the existence of a sequence $(v_n, g_n, A_{v_n g_n}, x_n)$ with

$$g_n \in Jv_n, \quad x_n \in S_X, \quad \|A_{v_n g_n}x_n\| = \|A_{v_n g_n}\|, \quad u \prec A_{v_n g_n}x_n \prec v_n,$$

that converges to $(\bar{v}, g, A_{\bar{v}g}, x)$ satisfying

$$(v_n) \downarrow \bar{v}, \quad g \in J\bar{v}, \quad x \in S_X, \quad u \preceq A_{\bar{v}g}x \preceq \bar{v}.$$

If, for every $z \in S_X$ such that $u \preceq A_{\bar{v}g}z \preceq \bar{v}$, $A_{\bar{v}g}$ does not attain its norm at z, then there exists $y \in S_X$ such that

$$\|A_{\bar{v}g}x\| < \|A_{\bar{v}g}y\| = \|A_{\bar{v}g}\|, \quad \bar{v} \prec A_{\bar{v}g}y \prec -u.$$

Hence, for n large enough $\|A_{v_n g_n}x_n\| < \|A_{v_n g_n}y\|$, which is contradictory. Therefore

$$\exists x \in S_X, \exists g \in J\bar{v} : \|A_{\bar{v}g}x\| = \|A_{\bar{v}g}\|, \quad u \prec A_{\bar{v}g}x \prec \bar{v}.$$

(2) Analogous arguments show that

$$\exists y \in S_X, \exists g \in J\bar{v} : \|A_{\bar{v}g}y\| = \|A_{\bar{v}g}\|, \quad \bar{v} \prec A_{\bar{v}g}y \prec -u.$$

(3) We have seen in (1) and (2) that there exist $g_1, g_2 \in J\bar{v}$ and $x, y \in S_X$ such that

$$\|A_{\bar{v}g_1}x\| = \|A_{\bar{v}g_1}\|, \quad u \prec A_{\bar{v}g_1}x \prec \bar{v},$$
$$\|A_{\bar{v}g_2}y\| = \|A_{\bar{v}g_2}\|, \quad \bar{v} \prec A_{\bar{v}g_2}y \prec -u.$$

Since, for every $0 \leqslant t \leqslant 1$, $tg_1 + (1-t)g_2 \in J\bar{v}$, a continuity argument similar to that used in (1) shows the existence of $\bar{g} = \bar{t}g_1 + (1-\bar{t})g_2$, $0 \leqslant \bar{t} \leqslant 1$, $\bar{x}, \bar{y} \in S_X$ such that

$$1 = \|A_{\bar{v}\bar{g}}u\| = \|A_{\bar{v}\bar{g}}\bar{v}\| < \|A_{\bar{v}\bar{g}}\bar{x}\| = \|A_{\bar{v}\bar{g}}\bar{y}\| = \|A_{\bar{v}\bar{g}}\|, \quad u \prec A_{\bar{v}\bar{g}}\bar{x} \prec \bar{v} \prec A_{\bar{v}\bar{g}}\bar{y} \prec -u.$$

Indeed, it suffices to take $\bar{t}$ as the infimum of the values of $t \in [0, 1]$ for which $A_{\bar{v}tg_1+(1-t)g_2}$ attains its norm at some $x \in S_X$ such that $u \prec A_{\bar{v}tg_1+(1-t)g_2}x \prec \bar{v}$. $\square$

Proposition 3.2. *If X is not an inner product space, then there exist $A, C \in \mathscr{L}(X)$ such that $A \perp C$, but there does not exist $x \in S_X$ such that $\|A\| = \|Ax\|$ and $Ax \perp Cx$.*

Proof. Let Y and A be as in Lemma 3.1. I.e., Y is a certain 2-dimensional subspace of X, $u, v \in S_Y$, $f \in Ju$, $g \in Jv$, and

$$A : y + z \in Y \oplus (\ker f \cap \ker g) \to y \in Y,$$

is a linear projection that satisfies

(i) $1 = \|Au\| = \|Av\| < \|A\|$.
(ii) The operator A attains its norm at points $x_1, x_2, x_3, x_4 \in S_X$ such that

$$u \prec Ax_1 \preceq Ax_2 \prec v \prec Ax_3 \preceq Ax_4 \prec -u.$$

(iii) If $x, y, z \in S_X$ are such that

$$u \prec Ax \prec Ax_1 \preceq Ax_2 \prec Ay \prec Ax_3 \preceq Ax_4 \prec Az \prec -u,$$

then

$$\|Ax\| < \|A\|, \quad \|Ay\| < \|A\|, \quad \|Az\| < \|A\|.$$

Let $C : X \to Y$ be the linear operator characterized by

$$Cx_1 = Ax_4 + Ax_1, \quad Cx_3 = Ax_2 - Ax_3, \quad Cz = 0 \quad \text{for } z \in \ker f \cap \ker g.$$

By definition of C and by convexity of $B(0, \|A\|) \cap Y = \{y \in Y : \|y\| \leqslant \|A\|\}$ we have that $Cx_3 \prec Cx_1$, or equivalently $Cx_3 \wedge Cx_1 > 0$. Note that C reverses the order with respect to A. That is,

$$Ax_1 \preceq Ax \preceq Ax_3 \quad \text{implies} \quad Cx_3 \preceq Cx \preceq Cx_1.$$

Indeed, let $Ax = \lambda Ax_1 + \mu Ax_3$ with $\lambda, \mu \geqslant 0$. Then

$$Cx_3 \wedge Cx = Cx_3 \wedge (\lambda Cx_1 + \mu Cx_3) = \lambda(Cx_3 \wedge Cx_1) \geqslant 0.$$

Therefore $Cx_3 \preceq Cx$. Now, it is easy to verify that the same arguments as in the proof of Proposition 2.5 show that $A \perp C$, and it follows there exists no $x \in S_X$ such that $Ax \perp Cx$. □

To sum up, we have proved

Theorem 3.3. *A real finite-dimensional normed space X is an inner product space if and only if, for $A, C \in \mathscr{L}(X)$,*

$$A \perp C \Leftrightarrow \exists x \in S_X : \|A\| = \|Ax\|, \quad Ax \perp Cx.$$

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